

# Vacuum stress-tensor in SSB theories

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The renormalized energy-momentum tensor of vacuum has been deeply explored many years ago. The main result of these studies was that such a tensor should satisfy the conservation laws which reflects the covariance of the theory in the presence of loop corrections. In view of this general result we address two important questions, namely how to implement the momentum cut-off in a covariant way and whether this general result holds in the theory with Spontaneous Symmetry Breaking. In the last case some new interesting details arise and although the calculations are more involved we show that the final result satisfies the conservation laws.

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## I. INTRODUCTION

Traditionally, the calculation of quantum corrections to the stress-tensor (also called Energy and Momentum Tensor - denoted as EMT in what follows) of vacuum is one of the most important issues of Quantum Field Theory in curved space-time. The reasons for the special interest in this problem are becoming obvious if we remember that the matter fields and particles enter the cosmological and most other gravitational equations in the form of EMT of matter, which is usually taken as a fluid. The quantum effects of field fluctuations turn out to give some corrections to the corresponding equations of state. The most relevant example, probably, is that the EMT for radiation gains a non-zero trace due to the conformal (trace) anomaly, which really changes the

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equation of state for the radiation with some possible relevant effects for the radiation-dominated Universe [1]. Opposite to this case, the equation of state for the massive particles and baryonic matter in general, does not change essentially, because quantum corrections can not make such a matter content to be relativistic.

The situation is quite different in the case of vacuum quantum effects, which can be much more relevant than those of the matter sector. Recently there were many publications on this subject, including the ones where the possible quantum effects of quantum massive matter fields on cosmology and astrophysics were explored. In particular, it was noticed in [2–7] that such quantum corrections can be defined up to a single free parameter  $\nu$  on the background of general covariance. Some observational consequences of the possible quantum corrections were explored in [8, 9] and led to establishing an upper bound on the magnitude of  $\nu$ . Furthermore, the same unique form of quantum corrections was applied also to astrophysics [5] and was shown [10] to provide an accurate description of the rotation curves for some sample set of disk galaxies, without introducing a large amount of Dark Matter content (see [11] for more examples). Some other applications to cosmology and astrophysics were also discussed in Ref. [12].

The applications mentioned above are based on a single, however nontrivial assumption of the existence of relevant quantum corrections in the low-energy vacuum sector. Needless to say that the most desirable development would be to derive such quantum effects on the regular basis in the framework of some rigorous QFT approach. The problem was discussed in [13] and the final conclusion concerning existing regular methods was essentially negative. The required quantum correction to the effective action of vacuum should be given by a sum of infinite products of the curvature tensor components with an infinite number of non-local insertions, hence there are small chances for a practical realization of such a calculus. One can note that the situation becomes much more definite if we give up the covariance and use, for instance, the conformal parametrization of the background metric. In this case it is possible to calculate the quantum corrections [14]. However, this method is not really safe and is anyway applicable only at the high-energy regime when the minimal subtraction procedure is supposed to be reliable.

It would be very nice to have some alternative approach to the derivation of desirable quantum corrections. Recently there were some publications where the result was obtained by means of the cut-off regularization in the conformally flat cosmological metric case [15] (early version) and [16] (see also [17]). The main idea is to perform calculations of the “energy density” and “pressure” of the vacuum in the momentum cut-off regularization, taking the expansion of the Universe into account perturbatively, order by order in the Hubble parameter  $H$ . The zero-order approximation

has been considered before by Akhmedov in [18] and earlier by DeWitt [19]. The output of the non-covariant procedure is not the naively expected equation “equation of state”  $p_{vac} = -\rho_{vac}$  of the cosmological constant, but the one for the radiation  $p_{vac} = \rho_{vac}/3$ , which led to several attempts to understand this result and even to correct it at the *ad hoc* basis [20]. In fact, DeWitt explained the result in a very general terms as being produced by the non-covariant regularization. The calculations of [15, 16] were based on the subtraction of the flat-space result of [18], which led to the new “equation of state” for the vacuum, this time proportional to  $H^2$  times the square of the cut-off parameter. The main problem with this result is that it apparently contradicts either the general covariance of the effective action, or the locality of the requested counterterms. In this case we meet a violation of the well established fundamental features of renormalization in curved-spaces (see, e.g., books [21, 22] and recent papers [23]). However, the results of these calculations should be considered as a motivation for the study of possible existence of the  $\mathcal{O}(H^2)$ -type corrections to the vacuum energy in cosmology. At the same time it looks very important to better understand these results at the technical level. This consideration is one of the motivations for the present paper. Furthermore, it is interesting to see how the calculations in the cut-off regularization can be done covariant. This problem has been recently solved in [24] on the basis of local momentum representation in Riemann normal coordinates (alternatively, one can achieve the covariance of finite expressions by imposing the conservation law step by step when adding specially adjusted non-covariant counterterms [15, 25]). Furthermore, there is one more possibility which deserves to be checked in full details. The cosmological constant term consists of the two main contributions [26], namely the vacuum classical term and the induced term. The no-go statement of [13] concerns only the quantum contribution to the vacuum part and, therefore, there is a chance to meet  $\mathcal{O}(H^2)$ -type quantum corrections to the vacuum energy from induced part. As one can see in what follows, for the induced contribution the route from effective action to the EMT is not so direct as it is for the vacuum counterpart. The corresponding calculation requires more efforts and concerns the main purpose of the present paper. We shall derive the quantum contribution to EMT from the induced term in the covariant way, in the linear in curvature approximation and will eventually show that in this approximation EMT of vacuum is local, satisfies the conservation law and hence it is given by a linear combination of the metric and Einstein tensor.

The paper is organized as follows. In Sect. 2 we present a brief summary of renormalization in curved space-time and discuss the non-covariant results obtained on the cosmological background from this perspective. In Sect. 3 we consider, following [27], the spontaneous symmetry breaking in curved space and derive the corresponding classical vacuum EMT in the linear in curvature

approximation. Sect. 4 is devoted to the conservation law for the EMT of the vacuum in the theories with SSB. In Sect. 5 we present an additional technical discussion of the classical EMT of the vacuum and its physical relevance in different theories. In Sect. 6 we derive the one-loop quantum correction to this EMT. Finally, in Sect. 7 we draw our conclusions and present some additional discussions. Some calculations concerning the normal coordinates and local momentum representation are addressed in Appendix A and a detailed derivation of equations of motion in the linear in curvature approximation is contained in Appendix B.

## II. BRIEF SUMMARY OF RENORMALIZATION IN CURVED SPACE

The renormalization of quantum theory of matter fields in curved space-time was subject of many investigations starting from [28]. The most simple way to remove divergences by the consistent renormalization procedure is related to the effective action method [22, 29, 30] (including by means of Batalin-Vilkovisky formalism [23]). The result of all these studies can be formulated in a simple form as follows: the theory of quantum matter fields which is renormalizable in flat space can be formulated as renormalizable in curved space if there is a regularization which is consistent with general covariance from one side and the gauge symmetries of the theory from another one. The renormalizability means that the divergences of effective action (at any loop order) are local and general covariant expressions compatible with the given gauge symmetries.

From the effective action perspective the renormalization of EMT is looking quite trivial: one has to derive the effective action  $\Gamma$  and take the variational derivative

$$\langle T_{\mu\nu}(x) \rangle = - \frac{2}{\sqrt{-g(x)}} g_{\mu\alpha}(x) g_{\nu\beta}(x) \frac{\delta\Gamma}{\delta g_{\alpha\beta}(x)}. \quad (1)$$

After that one has to introduce the counterterms into the effective action and add them to the  $\Gamma$  in (1), which is equivalent to performing some very special subtraction of the divergent terms. This subtraction should exactly correspond to the covariant and local counterterms in the effective action. After that the coefficients of the remaining finite terms should be fixed by imposing the renormalization conditions on the renormalized classical action and/or renormalized EMT. For this end such a classical action should be chosen in a special way and include all the structures which are possible to emerge as counterterms.

The arguments based on covariance, locality and power counting lead to the following form of the classical action of external metric (vacuum):

$$S_{vac} = S_{EH} + S_{HD}, \quad (2)$$

where  $S_{EH}$  is the Einstein-Hilbert action with the cosmological constant

$$S_{EH} = -\frac{1}{16\pi G} \int d^4x \sqrt{-g} (R + 2\Lambda) . \quad (3)$$

and

$$S_{HD} = \int d^4x \sqrt{-g} \{a_1 C^2 + a_2 E + a_3 \square R + a_4 R^2\} . \quad (4)$$

Here  $C^2 = R_{\mu\nu\alpha\beta}^2 - 2R_{\alpha\beta}^2 + (1/3) R^2$  is the square of the Weyl tensor and  $E = R_{\mu\nu\alpha\beta}^2 - 4R_{\alpha\beta}^2 + R^2$  is the integrand of the Gauss-Bonnet topological term (Euler density in  $d = 4$ ). Let us remark that the presence of higher derivative terms and cosmological constant are necessary to have a renormalizable theory.

In the present paper we will be interested to perform covariant calculations around the flat space-time in the linear in curvature approximation. This means we will systematically ignore the higher derivative part (4) and, in general, will not pay attention to the  $\mathcal{O}(R_{\dots}^2)$  and  $\mathcal{O}(\square R)$ -terms. This means, in particular, that the form of the divergent structures which one can meet in  $\langle T_{\mu\nu} \rangle$  is restricted to the two terms, namely the ones proportional to  $g_{\mu\nu}$  which are responsible for the renormalization of the cosmological constant term and the ones proportional to the Einstein tensor  $G_{\mu\nu} = R_{\mu\nu} - (1/2)Rg_{\mu\nu}$  and responsible for the renormalization of the Einstein-Hilbert term in the effective action.

Our calculations will be performed in the local momentum representation, based on the use of Riemann normal coordinates. Also, we shall use very simple cut-off regularization in the Euclidean local momentum space. This regularization has been shown equivalent to the cut-off of the proper time integral in the Schwinger formalism in flat space [31] and recently has been used in [32] to calculate effective potential of the scalar field in curved space-time.

An alternative approach to renormalize EMT in curved space-time is to work directly with the classical expression for the EMT and perform calculation. This approach is the most traditional one (see [21] and references therein). The covariant calculations in this way have been performed in [33] and [34] by means of the point-splitting regularization, without or with the use of effective action method. The covariant structure of divergences of EMT which has been described before is restored in the limit of zero splitting, but only if this limit is taken in a special invariant way.

Let us consider the result of [33] for the quartic divergent part of the quantum corrections to EMT. For the sake of simplicity we can deal with the flat space expressions, because the quartic divergences are not really affected by this choice. Then

$$\langle T_{\mu\nu} \rangle_{\text{quart. div}} = \frac{1}{2\pi^2} \frac{1}{n_\alpha n^\alpha} \left( g_{\mu\nu} - 4 \frac{n_\mu n_\nu}{n_\beta n^\beta} \right), \quad (5)$$

where  $n_\alpha$  is a small non-null four-vector defining the point splitting regularization of the corresponding Green functions  $G(x, x) \rightarrow G(x, x + n)$ . Now, if we chose the vector  $n$  in temporal direction  $n = (\epsilon^2, 0.0.0)$  we get that

$$\langle T_{\mu\nu} \rangle_{\text{quart. div}} = -\frac{1}{2\pi^2\epsilon^4} \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (6)$$

which is a traceless quartic divergent component of the total energy-momentum tensor. This result is in accordance with the one based on naive momentum cut-off in flat space of [18]. As it was explained in [19], there is no contradiction with the expected local Lorentz invariance of the divergences, because the origin of the (6) is in the use of a non-covariant regularization. In case of the point-splitting with temporal direction the breaking of Lorentz invariance is due to the non-relativistic choice  $n = (\epsilon^2, 0.0.0)$ . In case of cut-off regularization the origin of a non-covariance is different but since it is equally non-relativistic, the final result is the same.

One expects that a Lorentz invariant regularizations would give rise to [33] [34]

$$\langle T_{\mu\nu} \rangle_{\text{quart. div}} = \frac{1}{2\pi^2\epsilon^4} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad (7)$$

which is proportional to the Minkowski metric tensor and can be interpreted as a standard divergent contribution to the zero point energy or cosmological constant term.

Let us show how this occurs in Pauli-Villars regularization [35]. The contribution of a free scalar field with mass  $m$  to the vacuum energy  $\rho$  is given by [18]

$$\rho = \frac{1}{16\pi^2} \left( \Omega^4 + m^2\Omega^2 + \frac{1}{8}m^4 - \frac{1}{2}m^4 \log \frac{2\Omega}{m} + \mathcal{O}\left(\frac{m}{\Omega}\right) \right), \quad (8)$$

where  $\Omega$  is a 3-momentum space cut-off. Whereas, the same contribution to the pressure reads,

$$p = \frac{1}{48\pi^2} \left( \Omega^4 - m^2\Omega^2 - \frac{7}{8}m^4 + \frac{3}{2}m^4 \log \frac{2\Omega}{m} + \mathcal{O}\left(\frac{m}{\Omega}\right) \right). \quad (9)$$

The Pauli-Villars regularization is defined by a family of scalar and ghost fields with masses  $m_i^2 = \mu_i^2 M^2 + m^2, i = 1, 2, \dots, N$  with degeneracies  $s_i$ . Positive degeneracies correspond to scalar fields and negative degeneracies to ghost fields. The Pauli-Villars conditions

$$\sum_{i=1}^N s_i = -1, \quad \sum_{i=1}^N s_i \mu_i^2 = 0, \quad \sum_{i=1}^N s_i \mu_i^4 = 0 \quad (10)$$

guarantee that for a free field theory with mass  $m$  the quantum corrections to the vacuum energy and pressure are finite, i.e. all  $\Omega$  quartic, quadratic and logarithmic divergences are canceled out. Notice that there are always non-trivial solutions of the Pauli-Villars conditions equations (10), e.g.  $\mathbf{s} = (1, 1, -2, -1)$ ,  $\boldsymbol{\mu}^2 = (5, 8, 2, 9)$ . However, in the limit when the mass  $M$  of the Pauli-Villars regulators goes to infinity we recover the quartic divergences of the vacuum energy-momentum tensor which now are of the form

$$\langle T_{\mu\nu} \rangle_{\text{quart. div}} = M^4 t_{\mu\nu}^{(4)} = \frac{cM^4}{2\pi^2} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad (11)$$

where

$$c = -\frac{1}{16} \sum_{i=1}^N s_i \mu_i^4 \log \mu_i$$

is an arbitrary constant given in terms of the regulating Pauli-Villars parameters. Notice that the sign of the vacuum energy correction might become positive or negative depending of the choice of the regularization. In spite of the use of a non-covariant auxiliary cut-off the final result is covariant [36]. However, the existence of an ambiguity in the leading quartic divergence and its sign is a puzzling characteristic of quantum vacuum. From a renormalization viewpoint the ambiguities can be traced back to the locality of the cosmological constant term of the effective action. The effective value of the coupling has to be fixed by an explicit choice of renormalization prescription.

In the case of sub-leading divergences something similar occurs. The quadratic divergences also acquire a covariant form in Pauli-Villars regularization

$$\langle T_{\mu\nu} \rangle_{\text{quad. div}} = M^2 t_{\mu\nu}^{(2)} = \frac{c' M^2 m^2}{2\pi^2} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad (12)$$

with

$$c' = -\frac{1}{8} \sum_{i=1}^N s_i \mu_i^2 \log \mu_i. \quad (13)$$

In the case of quadratic divergences there is one more specific ambiguity, especially when they are calculated in the special cosmological background depending on the Hubble parameter  $H$ . Imagine we have obtained the result in the form

$$\langle T_{\mu\nu} \rangle = M^4 t_{\mu\nu}^{(4)} + M^2 H^2 t_{\mu\nu}^{(2)} + \dots \quad (14)$$

Now, in this expression  $H$  is effectively used as a constant, and therefore we can redefine the cut-off as  $M^2 \rightarrow M'^2 = M^2 + \lambda H^2$ , where  $\lambda$  is an arbitrary dimensionless parameter. As a result we arrive at the new form of the power-like divergence,

$$\langle T_{\mu\nu} \rangle = M^4 t_{\mu\nu}^{(4)} + M^2 H^2 [t_{\mu\nu}^{(2)} + 2t_{\mu\nu}^{(4)}] + \dots \quad (15)$$

with even greater degree of ambiguity. In the case of a theory of the quantum field with mass  $m$  one can perform a more general redefinition  $M^2 \rightarrow M'^2 = M^2 + \lambda H^2 + \tau m^2$ , with even more ambiguity, etc. It is important that the logarithmic divergences are not affected by this ambiguity and, in general, represent the most universal and well-defined part of quantum corrections [37] (see also [38] for a recent discussion of the subject and further references).

One simple way to get free of the mentioned ambiguities is by using the effective action method. The prescription which we have already described above is simple. First one has to derive the divergent and finite (at the level which is possible) of effective action, add counterterms, perform renormalization. At the second stage it is necessary to take a variational derivative with respect to the metric (1) and obtain the divergent part and/or renormalized EMT. In the next section we shall see that this procedure works even in the situation with SSB, where the procedure described above is essentially more complicated than in the free field case.

### III. SSB AND EMT IN CURVED SPACE

We start following Ref. [27]. However, since our purpose is to consider the most simple model with spontaneous symmetry breaking (SSB) in curved space, we will consider the single real scalar field, while in the mentioned reference the charged scalar was used. The classical action of the field  $\phi$  with a non-minimal coupling and a self-interaction is

$$S_{sc} = \int d^4x \sqrt{-g} \left\{ \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + \frac{1}{2} m^2 \phi^2 + \frac{1}{2} \xi R \phi^2 - \frac{\lambda}{4!} \phi^4 \right\}. \quad (16)$$

The dynamical equation for  $\phi$  has the form

$$-\square\phi + m^2\phi + \xi R\phi - \frac{1}{6}\lambda\phi^3 = 0. \quad (17)$$

Consequently, the vacuum expectation value (VEV) for the scalar field is then defined as solution of the equation

$$-\square v + m^2 v + \xi R v - \frac{1}{6}\lambda v^3 = 0. \quad (18)$$



It is easy to see that there is no constant solution for this equation for  $\xi \neq 0$ , while the value  $\xi = 0$  is inconsistent with renormalizability of the theory (see, e.g., [22]). Hence we can find the solution for the vacuum expectation value  $v$  only in the form of the power series in  $\xi R$ ,

$$v(x) = v_0 + v_1(x) + v_2(x) + \dots \quad (19)$$

In the zero-order approximation we meet the conventional flat-space expression,

$$v_0^2 = \frac{6m^2}{\lambda}. \quad (20)$$

As we have already mentioned above, in this paper we will use the approximation of small curvature and are interested in the first-order approximation only. At this level one can easily find a non-local expression

$$v_1 = \frac{\xi v_0}{\square + \lambda v_0^2/3} R. \quad (21)$$

In a similar way, it is possible to construct further approximations, but this is beyond the scope of the present paper. Thus, let us concentrate on the expression (21) and simplify it further by neglecting terms with derivatives of the scalar curvature. Of course, this approximation works only for the sufficiently large value of the square of the physical mass of the scalar excitation near the point of the minimum,  $2m^2 = \lambda v_0^2/3$ . Then we arrive at the quantity

$$v_1 \approx \frac{3\xi}{\lambda v_0} R. \quad (22)$$

It is clear that the same solution can be obtained directly from Eq. (18) if we disregard the term  $\square v$  and use (20). In our opinion the approach followed here is better, because it enables one to control the approximation. In further calculations we shall use the expression for classical solution of the theory in the point of the minima of the SSB problem,

$$\phi_{0c} = v = v_0 + v_1, \quad \text{where} \quad v_0^2 = \frac{6m^2}{\lambda} \quad \text{and} \quad v_1 = \frac{3\xi}{\lambda v_0} R. \quad (23)$$

The renormalization of the vacuum sector of the theory with SSB has been described in great detail in [27], thus, we shall not elaborate on it here. Instead, let us discuss the definition of the EMT at quantum level for the theory with SSB. The classical energy-momentum tensor of the field  $\phi$  in the external metric field  $g = g_{\mu\nu}$  is defined by the relation

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}} g_{\mu\alpha} g_{\nu\beta} \frac{\delta S[g, \phi]}{\delta g_{\alpha\beta}}. \quad (24)$$

At quantum level, the energy-momentum tensor is given by

$$\langle T_{\mu\nu} \rangle = - \frac{2}{\sqrt{-g}} g_{\mu\alpha} g_{\nu\beta} \left\langle 0 \left| \frac{\delta S[g, \hat{\phi}]}{\delta g_{\alpha\beta}} \right| 0 \right\rangle, \quad (25)$$

where  $\hat{\phi}$  is quantized field,  $\hat{\phi} \sim u\hat{a}^\dagger + u^*\hat{a}$  and  $\hat{a}|0\rangle = 0$ . As far as  $g_{\mu\alpha}$  is classical external field, so we can take it out of the  $\langle |..| \rangle$  freely.

According to our previous discussion, we will follow the functional representation of Quantum Field Theory, where the basic object is the generating functional of vertex function, or effective action,  $\Gamma = \Gamma[g, \phi]$ . For the case of a scalar field it is defined as a solution of the functional equation (see, e.g., [22] for a introduction)

$$\exp \left\{ \frac{i}{\hbar} \Gamma[g, \phi] \right\} = \int d\bar{\phi} \exp \left\{ \frac{i}{\hbar} \left( S[g, \bar{\phi} + \phi] - \frac{\delta \Gamma[g, \phi]}{\delta \phi} \bar{\phi} \right) \right\}. \quad (26)$$

In this work we restrict consideration by the one-loop approximation, when the effective action in Eq. (1) becomes the sum of the classical term and of the one-loop correction

$$\Gamma^{(1)}[\phi, g_{\mu\nu}] = S[\phi, g_{\mu\nu}] + \hbar \bar{\Gamma}^{(1)}[\phi, g_{\mu\nu}]. \quad (27)$$

Then the one-loop EMT of the vacuum can be cast into the form

$$\langle T_{\mu\nu}(x) \rangle^{(1)} = T_{\mu\nu}(x) + \bar{T}_{\mu\nu}^{(1)}(x), \quad (28)$$

where the first term is classical contribution,

$$T_{\mu\nu} = - \frac{2}{\sqrt{-g}} g_{\mu\alpha} g_{\nu\beta} \frac{\delta S}{\delta g_{\alpha\beta}} \Big|_{\phi \rightarrow \phi_0}. \quad (29)$$

and the second one is one-loop correction to it,

$$\bar{T}_{\mu\nu}^{(1)} = - \frac{2\hbar}{\sqrt{-g}} g_{\mu\alpha} g_{\nu\beta} \frac{\delta \bar{\Gamma}^{(1)}}{\delta g_{\alpha\beta}} \Big|_{\phi \rightarrow \phi_0}. \quad (30)$$

In both cases  $\phi_0$  is the solution of the equations of motion. If we deal with purely classical theory, then one has to replace in (29) the value  $\phi_0 = \phi_{0c}$  from Eq. (23). After the one-loop correction is taken into account, we have

$$\frac{\delta S[g, \phi_0]}{\delta \phi} + \hbar \frac{\delta \bar{\Gamma}^{(1)}[g, \phi_0]}{\delta \phi} = 0, \quad (31)$$

where the replacement  $\phi \rightarrow \phi_0$  should be performed after variational derivative. The Eq. (31) can be solved by iterations in  $\hbar$ . At one-loop level

$$\phi_0 = \phi_{0c} + \hbar \phi_1, \quad (32)$$

where  $\phi_{0c}$  is the classical solution (23). In order to find  $\phi_1$  one has to replace (32) into (31). In the first order in  $\hbar$  we meet the equation

$$\frac{\delta^2 S[g, \phi_{0c}]}{\delta\phi\delta\phi}\phi_1 + \frac{\delta\bar{\Gamma}^{(1)}[g, \phi_{0c}]}{\delta\phi} = 0, \quad (33)$$

and obtain the solution in the form

$$\phi_1 = - \left( \frac{\delta^2 S[g, \phi_{0c}]}{\delta\phi\delta\phi} \right)^{-1} \frac{\delta\bar{\Gamma}^{(1)}[g, \phi_{0c}]}{\delta\phi} \quad (34)$$

and, therefore,

$$\phi_0 = \phi_{0c} - \hbar \left( \frac{\delta^2 S[g, \phi_{0c}]}{\delta\phi\delta\phi} \right)^{-1} \frac{\delta\bar{\Gamma}^{(1)}[g, \phi_{0c}]}{\delta\phi}. \quad (35)$$

One has to replace this formula into the expression for EMT,

$$\langle T_{\mu\nu} \rangle = - \frac{2}{\sqrt{-g}} g_{\mu\alpha} g_{\nu\beta} \frac{\delta}{\delta g_{\alpha\beta}} \left\{ S[g, \phi_0] + \hbar \bar{\Gamma}^{(1)}[g, \phi_0] \right\}. \quad (36)$$

In this way we arrive at the general expression for the EMT in the scalar theory with SSB,

$$\begin{aligned} \langle T_{\mu\nu} \rangle = & - \frac{2}{\sqrt{-g}} g_{\mu\alpha} g_{\nu\beta} \left\{ \frac{\delta S[g, \phi_{0c}]}{\delta g_{\alpha\beta}} \right. \\ & \left. + \hbar \frac{\delta\bar{\Gamma}^{(1)}[g, \phi_{0c}]}{\delta g_{\alpha\beta}} - \hbar \frac{\delta^2 S[g, \phi_{0c}]}{\delta g_{\alpha\beta} \delta\phi} \left( \frac{\delta^2 S[g, \phi_{0c}]}{\delta\phi\delta\phi} \right)^{-1} \frac{\delta\bar{\Gamma}^{(1)}[g, \phi_{0c}]}{\delta\phi} \right\}. \end{aligned} \quad (37)$$

The first term inside the brackets is classical, the second is typical for the free theory and actually does not depend too much on the kind of such theory. The last term emerges only due to the fact that we deal with the interacting theory. In the free theory this term is zero.

It proves useful to define

$$\langle T_{\mu\nu} \rangle = \langle T_{\mu\nu} \rangle_v + \langle T_{\mu\nu} \rangle_i, \quad (38)$$

where

$$\langle T_{\mu\nu} \rangle_v = - \frac{2}{\sqrt{-g}} g_{\mu\alpha} g_{\nu\beta} \left\{ \frac{\delta S[g, \phi_{0c}]}{\delta g_{\alpha\beta}} + \hbar \frac{\delta\bar{\Gamma}^{(1)}[g, \phi_{0c}]}{\delta g_{\alpha\beta}} \right\} \quad (39)$$

and

$$\langle T_{\mu\nu} \rangle_i = \frac{2\hbar}{\sqrt{-g}} g_{\mu\alpha} g_{\nu\beta} \left\{ \frac{\delta^2 S[g, \phi_{0c}]}{\delta g_{\alpha\beta} \delta\phi} \left( \frac{\delta^2 S[g, \phi_{0c}]}{\delta\phi\delta\phi} \right)^{-1} \frac{\delta\bar{\Gamma}^{(1)}[g, \phi_{0c}]}{\delta\phi} \right\}. \quad (40)$$

The quantities  $\langle T_{\mu\nu} \rangle_v$  and  $\langle T_{\mu\nu} \rangle_i$  represent the vacuum and induced parts of the EMT, respectively. Both quantities will be calculated in this work in the context of SSB.

#### IV. COVARIANCE AND CONSERVATION OF VACUUM EMT

The conservation of the energy-momentum tensor of vacuum is always regarded as the main requirement for the consistency of the theory (see, e.g., [21] and further references therein). As far as we deal with the lower-derivative approximation, the condition of conservation  $\nabla^\mu \langle T_{\mu\nu} \rangle = 0$ , together with the requirement that the EMT should be derived as variational derivative of covariant effective action, can fix the algebraic form of  $\langle T_{\mu\nu} \rangle$  completely, leaving the room for only two numerical parameters in case of quadratic and logarithmic divergences and finite part and for a single numerical parameter for the quartic divergences case.

The reasons for this special importance of the conservation law are as follows. For the divergent parts of effective action the situation is especially simple, because we know it should be local (see [23] for a recent discussion of this issue in curved space-time). As we have already explained in the previous section, in the lower-derivative sector this means that the possible counterterms have the form of the Einstein-Hilbert term and of the cosmological constant term (3). Consequently, the divergent part of the vacuum EMT should consist of only two structures, namely

$$\langle T_{\mu\nu} \rangle = C_1 g_{\mu\nu} + C_2 G_{\mu\nu}, \quad (41)$$

where  $C_1 = k_4 \Omega^4 + k_2 \Omega^2 + k_L \ln(\Omega/\mu_0)$  and  $C_2 = l_2 \Omega^2 + l_L \ln(\Omega/\mu_0)$ , with  $k_4, k_2, k_L$  and  $l_2, l_L$  being numerical constants. The values of these constants depend on the choice of the quantum theory, on the order of loop expansion, but the structure of divergent part must be always like in (41).

Concerning the finite part of the EMT of the vacuum, it is possible to have much more complicated expression than the one presented in (41), as a result of resummation of the series in curvatures and Green functions [13]. One can have an indication to this possibility, e.g., from the calculation in conformal variables [14]. However, as far as we are going to perform a relatively simple calculation in the  $\mathcal{O}(R)$ -approximation, there is no room for non-localities in the effective action, so what one should expect as a result is the same expression (41).

Needless to say that (41) is the only form of EMT which can be derived from some action principle and also is the only form which satisfies conservation law. Let us start from a well-known derivation of this relation in general case and then consider the same thing in view of Eq. (37).

The effective action  $\Gamma$  is covariant scalar functional depending on metric  $g_{\mu\nu}$  and scalar field  $\phi$ . If we perform infinitesimal general coordinate transformation  $x^\alpha \rightarrow x'^\alpha = x^\alpha + \xi^\alpha(x)$ , these two

fields transform according to the known rules

$$\delta g_{\mu\nu} = -\nabla_\mu \xi_\nu - \nabla_\nu \xi_\mu, \quad \delta\phi = -\xi^\mu \partial_\mu \phi. \quad (42)$$

Then the identity corresponding to diffeomorphism invariance of  $\Gamma$  is

$$\int d^4x \sqrt{-g} \left\{ \frac{2}{\sqrt{-g}} \frac{\delta\Gamma[g, \phi]}{\delta g_{\mu\nu}} \nabla_\mu \xi_\nu + \frac{1}{\sqrt{-g}} \frac{\delta\Gamma[g, \phi]}{\delta\phi} \xi^\mu \nabla_\mu \phi \right\} = 0. \quad (43)$$

Now we take into account that the functional derivative vanish on-shell, that means

$$\frac{\delta\Gamma[g, \phi_0]}{\delta\phi} = 0. \quad (44)$$

Then integrating the first term in (43) and taking into account the definition of EMT (1), we arrive at the conservation law,  $\nabla^\mu \langle T_{\mu\nu} \rangle_{\phi_0} = 0$ .

Let us now see how the same considerations look when we perform the expansion of  $\Gamma[g, \phi_0]$  into series in  $\hbar$ . Here we are interested in the expansion up to the first order and the main question is whether the two parts EMT, namely vacuum and induced ones, (40) and (39), do satisfy the conservation law separately or only when they are summed up.

At zero order everything is quite obvious, for we have

$$\frac{\delta S[g, \phi_{0c}]}{\delta\phi} = 0, \quad \text{and} \quad \nabla^\mu T_{\mu\nu}|_{\phi_{0c}} = 0. \quad (45)$$

At the first order in  $\hbar$  we notice that the conservation law is satisfied only on-shell. Now, the solution  $\phi_0 = \phi_{0c} + \hbar\phi_1$  of (34), was found exactly to provide that  $\phi_0$  is the solution of the effective equations of motion at one loop. Hence, we should expect that neither one of the two terms (40) and (39) will satisfy the conservation equation and only for their sum this equation must be valid,

$$\nabla^\mu \langle T_{\mu\nu}(\phi_{0c}) \rangle_v + \nabla^\mu \langle T_{\mu\nu}(\phi_{0c}) \rangle_i = 0. \quad (46)$$

On the other hand, this means that the sum (38) should have the form (41) while each term can have more arbitrary form, for example the Ricci tensor and scalar curvature term may not form Einstein tensor.

Finally, both the covariance arguments and conservation law indicate that the quantum EMT of vacuum, in the low-energy sector of the theory, must have the form (41) even in the presence of SSB which produce much more sophisticated forms of EMT, such as (37). The restricted form of the vacuum EMT (41) should hold even at higher loop orders, at least for divergent contributions[42]. This is a strong statement and it is worthwhile to check it by direct calculation, at least in the one-loop order. We shall do it in the next section.

## V. DERIVATION OF STRESS TENSOR: CLASSICAL PART

In this section we shall derive the EMT of vacuum at classical level and then, in the next sections, turn to the one-loop contributions.

Let us perform the calculation of the EMT  $T_{\mu\nu}$  in the vacuum state, which is characterized by the VEV of scalar field defined in Eq. (23). The calculation of  $T_{\mu\nu}$  is rather trivial and we obtain

$$\begin{aligned} T_{\mu\nu} = & -\frac{2}{\sqrt{-g}} g_{\mu\alpha} g_{\nu\beta} \frac{\delta S[g, \phi]}{\delta g_{\alpha\beta}} = \left(2\xi - \frac{1}{2}\right) g_{\mu\nu} (\nabla\phi)^2 + (1 - 2\xi) (\partial_\mu\phi) (\partial_\nu\phi) \\ & + 2\xi\phi(g_{\mu\nu}\square\phi - \nabla_\mu\nabla_\nu\phi) + \xi\phi^2\left(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}\right) + \frac{1}{2}g_{\mu\nu}m^2\phi^2 + \frac{\lambda}{24}g_{\mu\nu}\phi^4. \end{aligned} \quad (47)$$

The trace of the scalar EMT on-shell (17) can be easily reduced to the form

$$T^\mu_\mu = (6\xi - 1) \left[ (\nabla\phi)^2 + R\phi^2 - \frac{\lambda}{6}\phi^4 \right] + 2(1 - 3\xi) m^2\phi^2. \quad (48)$$

We observe that for  $m^2 = 0$  and  $\xi = 1/6$  we have  $T^\mu_\mu = 0$ . However, we are interested in the massive case given by Eq. (23). At this point it is worthwhile to discuss the practical realization of the  $\mathcal{O}(R)$  approximation, which we will follow in this section. The main question what to do with the derivatives of  $\phi_{0c} = v_0 + v_1$ . Since  $v_0$  is a constant, its derivative is obviously zero. Furthermore, a derivative of  $v_1$  gives us

$$\nabla_\alpha\phi_{0c} = \nabla_\alpha v_1 = \frac{3\xi}{\lambda v_0} \nabla_\alpha R \quad (49)$$

and, therefore, goes beyond the limits of our approximation. As a result we can always treat  $R$  and  $v_1$  as constants, that leads to great simplification of all calculation.

Replacing (23) into (47) and keeping only terms linear in curvature tensors, after small algebra we arrive at

$$T_{\mu\nu}(\phi_{0c}) = \xi v_0^2 \left( R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \right) - \frac{\lambda v_0^4}{12} g_{\mu\nu}. \quad (50)$$

This expression is nothing else but the induced contribution to the Einstein equations. It is natural to attribute it to the gravitational part of these equations, which can be, eventually, written as

$$\left( \frac{1}{8\pi G_{vac}} + \frac{1}{8\pi G_{ind}} \right) \left( R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \right) - \left( \rho_\Lambda^{vac} + \rho_\Lambda^{ind} \right) g_{\mu\nu} = T_{\mu\nu}^{matter}, \quad (51)$$

where

$$\frac{1}{8\pi G_{ind}} = -\xi v_0^2 \quad \text{and} \quad \rho_\Lambda^{ind} = -\frac{\lambda v_0^4}{12}. \quad (52)$$

In this equation  $G_{vac}$  and  $\rho_\Lambda^{vac}$  denote the vacuum Newton constant and the cosmological constant density, which are independent parameters that are originally present in the action of the theory.

Contrary to that,  $G_{ind}$  and  $\rho_{\Lambda}^{ind}$  are induced quantities which depend on the details of the quantum theory of matter fields under consideration.

The induced contributions here are due to the SSB, and an equivalent mechanism of their generation is working also for the Standard Model (SM) and Grand Unification Theories (GUT's). The values of induced and vacuum cosmological constants are known to be, at least, 55 orders of magnitude greater than their sum in (51), that gives rise to the cosmological constant problem [26] (see also references therein and [2] in relation to renormalization of the cosmological constant). On the contrary the relative magnitude of  $G_{ind}$ , namely

$$\frac{G_{ind}}{G_{vac}} = -\frac{8\pi\xi v_0^2}{M_P^2}, \quad (53)$$

is small for the SM case where  $v_0^2 \approx 10^5 GeV^2$ . Even if the value of  $\xi$  corresponds to the Higgs inflation,  $\xi \approx 4 \times 10^4$ , the Planck suppression is strong due to the relatively huge value  $M_P^2 \approx 10^{38} GeV^2$  and hence the contribution of (53) is irrelevant.

However, the situation can be quite different in GUT's, where (in the supersymmetric versions) we have  $v_0^2 \approx 10^{32} GeV^2$ . Then for the mentioned above magnitude of  $\xi$  we arrive at the estimate  $G_i/G_v \approx 1$  and the effective sum in (51) becomes close to zero. This means the value of  $1/G_v$  must be taken about twice larger than the observed sum. Hence, the classical screening due to induced value may be relevant in this case. In the rest of this paper we shall check that the quantum effects do not break the structure of (51) and calculate quantum corrections to the quantities  $G_{ind}$  and  $\rho_{\Lambda}^{ind}$  in (52).

## VI. ONE-LOOP CALCULATION IN THE $\mathcal{O}(R)$ -APPROXIMATION

Let us now perform quantum calculations using the expressions for vacuum and induced parts, (40) and (39). The calculations will be done in the local momentum representation and covariant momentum cut-off regularization. For better organization, this section is divided into subsections. First, we consider some general notions, then derive the flat-space result, then present some minimal mathematical tools for the local momentum representation, and finally perform derivation of the more complicated, curvature-dependent part.

### A. General considerations and derivation of $\langle T_{\mu\nu} \rangle_v$

Our starting point will be the one-loop effective action,  $\bar{\Gamma}^{(1)}[g, \phi]$ . By construction, this is the effective action in the theory with unbroken symmetry. One can write, using derivative expansion,

$$\bar{\Gamma}^{(1)}[g, \phi] = \int d^4x \sqrt{-g} \left\{ -\bar{V}_{eff}(\phi) + \frac{1}{2} \nabla_\mu \phi \cdot k_\phi \left( \frac{\square}{m^2} \right) \nabla^\mu \phi + \frac{1}{2} \phi^2 k_\xi \left( \frac{\square}{m^2} \right) R + \dots \right\}, \quad (54)$$

where the effective potential part has the form

$$\bar{V}_{eff}(\phi) = V_0 + V_1 R + \mathcal{O}(R^2), \quad (55)$$

which was recently calculated using covariant momentum cut-off in [32] and  $k_\phi \left( \frac{\square}{m^2} \right)$  and  $k_\xi \left( \frac{\square}{m^2} \right)$  are the form factors which also contain different powers of derivatives. The expansion in (54) is infinite, but we can easily set the limit on it, following the same approach which was used in the previous section. For  $\xi = 0$  we know  $\phi_{0c} = \text{const}$ , according to Eq. (17). Therefore, any derivatives of  $\phi_{0c}$  are actually proportional to  $\xi$  and hence to  $R$ .

As far as we are interested only in  $\mathcal{O}(R)$ -terms, we can take only constant part of the form-factor  $k_\xi$  in (54), and also strongly restrict  $k_\phi$  form factor, also by taking its constant part. Hence we can trade

$$k_\phi \left( \frac{\square}{m^2} \right) \rightarrow Z(\phi), \quad \text{and} \quad k_\xi \left( \frac{\square}{m^2} \right) \rightarrow \chi(\phi). \quad (56)$$

Futhermore, in the given approximation the term  $\frac{1}{2} \phi^2 \chi(\phi) R$  is a part of the effective potential  $V_1 = V_1(\phi)$ . So, for us  $\bar{\Gamma}^{(1)}[g, \phi]$  becomes

$$\bar{\Gamma}^{(1)}(g, \phi) = \int d^4x \sqrt{-g} \left\{ -\bar{V}_{eff}(\phi) + \frac{1}{2} Z(\phi) (\nabla \phi)^2 \right\}, \quad (57)$$

with  $\bar{V}_{eff} = V_0(\phi) + V_1(\phi) R$  and  $\phi \rightarrow \phi_{0c}$ .

Let us consider

$$\begin{aligned} \int d^4x \sqrt{-g} Z(\phi) (\nabla \phi)^2 &= \int d^4x \sqrt{-g} \nabla_\mu \chi^\mu - \int d^4x \sqrt{-g} Z(\phi) \phi \square \phi \\ &\quad - \int d^4x \sqrt{-g} Z'(\phi) (\nabla \phi)^2. \end{aligned} \quad (58)$$

For  $\phi \rightarrow \phi_{0c}$ , the quantity  $\square \phi$  can be written as

$$\square \phi_{0c} = \square(v_0 + v_1) = \frac{\xi v_0}{\square + 2m^2} \square R = \frac{\xi v_0}{2m^2} \square R + \mathcal{O}(\square^2 R). \quad (59)$$

On the other hand,

$$\nabla_\mu \phi_{0c} = \nabla_\mu v_0 + \nabla_\mu v_1 = \nabla_\mu v_1 = \frac{\xi v_0}{2m^2} \nabla_\mu R + \mathcal{O}(\nabla^3 R). \quad (60)$$



It is now obvious that, because of  $\square \ll m^2$  for  $\phi_{0c}$ , the whole quantity  $(\nabla\phi)^2$  is beyond our approximation  $\mathcal{O}(R)$ . Finally, we can restrict our consideration by the effective potential, using the expression

$$\bar{\Gamma}^{(1)}[g, \phi_{0c}] = - \int d^4x \sqrt{-g} \bar{V}_{eff}(\phi_{0c}). \quad (61)$$

The renormalized expression of the potential is [32]

$$\begin{aligned} \bar{V}_{eff}^{ren}(g_{\mu\nu}, \varphi) &= V_0^{ren} + V_1^{ren} R \\ &= \frac{1}{2(4\pi)^2} \left[ \frac{1}{2} (V'' - m^2)^2 - \left( \xi - \frac{1}{6} \right) R (V'' - m^2) \right] \ln \left( \frac{V'' - m^2}{\mu^2} \right). \end{aligned} \quad (62)$$

In this expression we used a general form of classical interaction term  $V = V(\varphi)$ , but later on it will be replaced by  $V = \lambda\varphi^4/4$

For the sake of completeness we will also consider the divergent part of the non-renormalized potential, in the local momentum cut-off regularization. In the given approximation we have  $\bar{V}_{eff}^{div}(g_{\mu\nu}, \varphi) = V_0^{div} + V_1^{div} R$ , where

$$\bar{V}_0^{div} = \frac{1}{32\pi^2} \left\{ \Omega^2 V'' - \frac{1}{2} (V'' - m^2)^2 \ln \frac{\Omega^2}{m^2} \right\}, \quad (63)$$

$$\bar{V}_1^{div} = \frac{1}{32\pi^2} \left( \xi - \frac{1}{6} \right) \left\{ -\Omega^2 + (V'' - m^2) \ln \frac{\Omega^2}{m^2} \right\}, \quad (64)$$

It proves useful to introduce a notation for the one-loop contributions to the equations of motion for a scalar field,

$$\bar{\varepsilon}^{(1)} = \bar{\varepsilon}_{div}^{(1)} + \bar{\varepsilon}_{fin}^{(1)} = \frac{1}{\sqrt{-g}} \frac{\delta \bar{\Gamma}^{(1)}}{\delta \phi} \Big|_{\phi_{0c}} = - \frac{\partial \bar{V}_{eff}^{(1)}}{\partial \phi} \Big|_{\phi_{0c}}. \quad (65)$$

After adding the corresponding counterterm, we will also have  $\bar{\varepsilon}_{ren}^{(1)}$ . Let us now remember that  $\bar{V}_{eff} = V_0(\phi) + V_1(\phi) R$ ,  $\phi_{0c} = v_0 + v_1$ , and define also

$$\begin{aligned} \bar{\varepsilon}^{(1)} &= - \frac{\partial \bar{V}_0^{(1)}}{\partial \phi} \Big|_{\phi_{0c}} - R \frac{\partial \bar{V}_1^{(1)}}{\partial \phi} \Big|_{\phi_{0c}} \\ &= - \frac{\partial \bar{V}_0^{(1)}}{\partial \phi} \Big|_{v_0} - \frac{\partial^2 \bar{V}_0^{(1)}}{\partial \phi^2} \Big|_{v_0} v_1 - R \frac{\partial \bar{V}_1^{(1)}}{\partial \phi} \Big|_{v_0} = \bar{\varepsilon}_0^{(1)} + \bar{\varepsilon}_1^{(1)}. \end{aligned} \quad (66)$$

Obviously, both  $\bar{\varepsilon}_0^{(1)}$  and  $\bar{\varepsilon}_1^{(1)}$  have finite and divergent parts and after adding counterterms we can also define their renormalized versions. In this paper we will calculate divergent and renormalized quantities only, but the original finite parts can be calculated in the same way using, e.g., the effective potential from [32]. The last relevant observation is that, within the  $\mathcal{O}(R)$  approximation adopted here we can treat all versions of  $\bar{\varepsilon}_0^{(1)}$  and  $\bar{\varepsilon}_1^{(1)}$  (divergent, finite non-renormalized, counterterms and renormalized) as constants.

Let us now calculate the simplest quantum term  $\langle T_{\mu\nu} \rangle_v$ , defined in (39). Starting from (61) we can easily arrive at

$$\begin{aligned}
\langle \bar{T}_{\mu\nu} \rangle_v &= -\frac{2\hbar}{\sqrt{-g}} g_{\mu\alpha} g_{\nu\beta} \frac{\delta \bar{\Gamma}^{(1)}[g, \phi_{0c}]}{\delta g_{\alpha\beta}} \\
&= -2\hbar V_1(v_0) \left( R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \right) + \hbar V_0(v_0) g_{\mu\nu} + \hbar v_1 g_{\mu\nu} \frac{\partial \bar{V}_0^{(1)}}{\partial \phi} \Big|_{v_0} \\
&= -2\hbar V_1(v_0) G_{\mu\nu} + \hbar V_0(v_0) g_{\mu\nu} - \frac{\hbar \xi v_0}{2m^2} R \bar{\varepsilon}_0^{(1)} g_{\mu\nu}.
\end{aligned} \tag{67}$$

This formula is remarkable, because it confirms what we have anticipated in the previous section. The first two terms in the last expression are quantum contributions to the Einstein tensor and cosmological constant part in the Einstein equations. However, the last term looks odd, for it violates covariance, conservation law and can not be derived from the action principle. So, we should hope that it will cancel with the corresponding contribution from  $\langle \bar{T}_{\mu\nu} \rangle_i$  in (40), which is the last term in (37). Let us see whether this really happens in the next section.

### B. Calculation of $\langle \bar{T}_{\mu\nu} \rangle_i$

Our first step will be to rewrite the expression (40) for  $\langle \bar{T}_{\mu\nu} \rangle_i$  in a more useful and detailed form

$$\begin{aligned}
\langle T_{\mu\nu}(x) \rangle_i &= 2\hbar g_{\mu\alpha}(x) g_{\nu\beta}(x) \int d^4y \sqrt{-g(y)} \int d^4z \sqrt{-g(z)} \left( \frac{1}{\sqrt{-g}} \frac{\delta^2 S[g, \phi_{0c}]}{\delta g_{\alpha\beta}(x) \delta \phi(y)} \right) \\
&\quad \times \left( \frac{1}{\sqrt{-g(y)}} \frac{\delta^2 S[g, \phi_{0c}]}{\delta \phi(y) \delta \phi(z)} \right)^{-1} \times \left( \frac{1}{\sqrt{-g(z)}} \frac{\delta \bar{\Gamma}^{(1)}[g, \phi_{0c}]}{\delta \phi(z)} \right).
\end{aligned} \tag{68}$$

Let us note that the metric-dependent quantities are always understood through the normal coordinate expansions (see some details of this technique in Appendix A).

The next step is to derive all three factors inside the integrals of the Eq. (68). We need to perform this calculation in the  $\mathcal{O}(R)$  approximation, which we follow here. The first factor can be obtained by varying (47) with respect to  $\phi$  or just taking a second variation of the action. After some algebra we arrive at

$$\begin{aligned}
\frac{1}{\sqrt{-g}} \frac{\delta^2 S}{\delta \phi(y) \delta g_{\mu\nu}(x)} &= \xi \phi (\nabla_\mu \nabla_\nu - g_{\mu\nu} \square) + (2\xi - 1) (\nabla_\mu \phi) \nabla_\nu \\
&\quad + \left( \frac{1}{2} - 2\xi \right) g_{\mu\nu} (\nabla^\lambda \phi) \nabla_\lambda + \xi (\nabla_\mu \nabla_\nu \phi - g_{\mu\nu} \square \phi) \\
&\quad - \xi \phi \left( R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \right) + \frac{1}{2} m^2 \phi g_{\mu\nu} - \frac{\lambda}{6} \phi^3 g_{\mu\nu}.
\end{aligned} \tag{69}$$

Now one can replace in the last expression  $\phi \rightarrow \phi_{0c} = v_0 + v_1$  and remember that all derivatives

of  $\phi_{0c}$  are beyond our approximation. In this way we obtain

$$\begin{aligned} \frac{1}{\sqrt{-g}} g_{\alpha\mu} g_{\beta\nu} \frac{\delta^2 S}{\delta g_{\alpha\beta} \delta \phi} \Big|_{\phi_{0c}} &= \xi \phi_{0c} (\nabla_\mu \nabla_\nu - g_{\mu\nu} \square) - \xi \phi_{0c} \left( R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \right) \\ &+ \frac{1}{2} m^2 \phi_{0c} g_{\mu\nu} - \frac{\lambda}{6} \phi_{0c}^3 g_{\mu\nu}. \end{aligned} \quad (70)$$

Finally, we replace here the expansion (19) up to the first order in curvature,  $\phi_{0c} = v_0 + v_1$ , with  $v_0$  and  $v_1$  taken from Eqs. (20) and (21). Also, we use the normal coordinates expansion of the operator  $\nabla_\mu \nabla_\nu - g_{\mu\nu} \square$  which is calculated in the Appendix A. The final result for the first factor inside the integral in Eq. (68) has the form

$$\begin{aligned} \frac{1}{\sqrt{-g}} \frac{\delta S}{\delta g_{\mu\nu} \delta \phi} \Big|_{\phi_{0c}} &= \xi v_0 (\partial_\mu \partial_\nu - \eta_{\mu\nu} \partial^2) + \frac{\xi^2 v_0}{2m^2} R (\partial_\mu \partial_\nu - \eta_{\mu\nu} \partial^2) - \xi v_0 R_{\mu\nu} \\ &+ \frac{1}{3} \xi v_0 \left[ 2 R^\lambda{}_{(\mu\nu)}{}^\tau y^\tau \partial_\lambda + 2 \eta_{\mu\nu} R^\lambda{}_\tau y^\tau \partial_\lambda + R_{\mu\alpha\nu\beta} y^\alpha y^\beta \partial^2 + \eta_{\mu\nu} R^\rho{}_{\alpha\beta}{}^\sigma y^\alpha y^\beta \partial_\rho \partial_\sigma \right]. \end{aligned} \quad (71)$$

The first term in the *r.h.s.* is of the zero order in curvature and the rest of the terms are of the first order in curvature. Indeed, after all calculations are completed, we will trade the metric  $\eta_{\mu\nu}$  in the point  $P$  to the general one  $g_{\mu\nu}$ , but for a while it is better we write it in the way we did.

Let us now consider the second factor inside the integral in Eq. (68),

$$\left( \frac{1}{\sqrt{-g}} \frac{\delta S^2[g, \phi_{0c}]}{\delta \phi \delta \phi} \right)_{y,z}^{-1} = G(y, z; \phi_{0c}), \quad (72)$$

This is nothing else but the propagator of the scalar excitations near the point of the minima. It is important to remember that we will need the dependence on the curvature. Therefore, according to [39, 40] (see also [32]) one has to modify the (72) to the form

$$\left( \frac{1}{[-g(y)]^{1/4} [-g(z)]^{1/4}} \times \frac{\delta S^2[g, \phi_{0c}]}{\delta \phi(y) \delta \phi(z)} \right)^{-1} = \bar{G}(y, z; \phi_{0c}). \quad (73)$$

Now we can use the known result for the propagator from the mentioned references [32, 39, 40], but first we have to evaluate the mass of the scalar excitations near the point of the minima.

One can start from the full propagator with  $\phi_{0c} = v_0 + v_1$ . Starting from the equation (17) we arrive at

$$\frac{1}{\sqrt{-g}} \frac{\delta^2 S}{\delta \phi \delta \phi} = -\square + m^2 + \xi R - \frac{\lambda}{2} \phi^2. \quad (74)$$

Next we replace

$$\phi \rightarrow \phi_{0c}^2 = (v_0 + v_1)^2 \approx v_0^2 + 2v_0 v_1. \quad (75)$$

Replacing (21) and (20) into (75), after some small algebra we arrive at

$$\begin{aligned} \frac{1}{\sqrt{-g}} \frac{\delta^2 S}{\delta\phi\delta\phi} &= -\square + 2m^2 + \xi \left(1 - \frac{6m^2}{\square + 2m^2}\right) R \\ &\approx -(\square + 2m^2) - 2\xi R, \end{aligned} \quad (76)$$

where at the last step we used the  $\mathcal{O}(R)$ -approximation, as it was already discussed above. Now, after we compare the last expression with Eq. (17), it is clear that (76) means we have a propagator of a scalar particle with *positive* mass  $2m^2$  and with the non-minimal parameter  $-2\xi$ . By using the general expression (107) we obtain the Euclidean version of the second factor inside the integral in Eq. (68) in the form

$$\bar{G}(z-y) = \int \frac{d^4 k}{(2\pi)^4} e^{ik(z-y)} \left[ \frac{1}{k^2 + 2m^2} - \left(2\xi - \frac{1}{6}\right) \frac{R}{(k^2 + 2m^2)^2} \right]. \quad (77)$$

The third factor of the integrand in Eq. (68) is nothing else but the effective equation of motion (65). According to (66) we can write it as a sum of classical and quantum parts,  $\varepsilon = \bar{\varepsilon}^{(0)} + \hbar\bar{\varepsilon}^{(1)}$ , where the last can be also expanded into series in scalar curvature,  $\bar{\varepsilon}^{(1)} = \bar{\varepsilon}_0^{(1)} + \bar{\varepsilon}_1^{(1)}$ . For the sake of completeness we have calculated these expressions, but since they are rather cumbersome, we postpone them to Appendix B.

Now we are in a position to derive  $\langle \bar{T}_{\mu\nu} \rangle_i$ . As a first step we obtain the flat-space expression and then consider a bit more complicated curvature-dependent terms.

In the flat-space limit we have only first terms in the *r.h.s.* of Eq. (71) and (77), and also need only  $\bar{\varepsilon}_0^{(1)}$ -parts in the equation of motion (both divergent and renormalized versions). In this way we arrive at the expression

$$\begin{aligned} \langle T_{\mu\nu}(x) \rangle_i^0 &= 2\hbar\xi v_0 \int d^4 z d^4 y \delta^4(x-y) \\ &\times (\partial_\mu \partial_\nu - \eta_{\mu\nu} \partial^2)_y \int \frac{d^4 k}{(2\pi)^4} \frac{e^{ik(y-z)}}{k^2 + 2m^2} \bar{\varepsilon}_0^{(1)}(z), \end{aligned} \quad (78)$$

where the upper index 0 indicates flat-space limit. After performing integration over  $y$  and using  $\bar{\varepsilon}_0^{(1)}(z) = \text{const}$ , we get

$$\begin{aligned} \langle T_{\mu\nu}(x) \rangle_i^0 &= 2\hbar\xi v_0 \int d^4 z (\partial_\mu \partial_\nu - \eta_{\mu\nu} \partial^2)_x \int \frac{d^4 k}{(2\pi)^4} \frac{e^{ik(x-z)}}{k^2 + 2m^2} \bar{\varepsilon}_0^{(1)}(z) \\ &= 2\hbar\xi v_0 \bar{\varepsilon}_0^{(1)} (\partial_\mu \partial_\nu - \eta_{\mu\nu} \partial^2)_x \int d^4 k \frac{e^{ikx}}{k^2 + 2m^2} \int \frac{d^4 z}{(2\pi)^4} e^{-ikz} \\ &= 2\hbar\xi v_0 \bar{\varepsilon}_0^{(1)} \int d^4 k \delta^4(k) \frac{k_\mu k_\nu - k^2 \eta_{\mu\nu}}{k^2 + 2m^2} e^{ikx} = 0. \end{aligned} \quad (79)$$

Thus, the contribution of the last term in (38) to the induced cosmological constant is zero.

As a first byproduct we also obtain that in curved space-time the contributions of the third factor, being it  $\bar{\varepsilon}_{1,div}^{(1)}$  or  $\bar{\varepsilon}_{1,ren}^{(1)}$ , are also vanishing. The reason is that both are constants in the  $\mathcal{O}(R)$  approximation and we did not use an explicit form of a constant  $\bar{\varepsilon}_0^{(1)}$  in the calculation presented above.

As a second byproduct we can see that in curved space-time the curvature-dependent contribution of the second factor (77) vanish too. The reason is that, if we trade

$$\frac{1}{k^2 + 2m^2} \rightarrow -\left(2\xi - \frac{1}{6}\right) \frac{R}{(k^2 + 2m^2)^2} \quad (80)$$

in (78), the zero output of the integral will obviously remain the same. So, after all we need to take into account only the curvature-dependent terms in the first factor, Eq. (71).

The last step is to perform the curved - space calculation in the  $\mathcal{O}(R)$  order. Taking into account the arguments presented above, we arrive at

$$\langle T_{\mu\nu}(x) \rangle_i^1 = 2\hbar \bar{\varepsilon}_0^{(1)} \int d^4y d^4z \int \frac{d^4k}{(2\pi)^4} \sum_{i=1}^5 O_{\mu\nu}^{(i)}(y) \delta^4(x-y) \frac{e^{ik(y-z)}}{k^2 + 2m^2}, \quad (81)$$

where

$$\begin{aligned} O_{\mu\nu}^{(1)} &= -\xi v_0 R_{\mu\nu}, \\ O_{\mu\nu}^{(2)} &= \frac{\xi^2 v_0}{2m^2} R (\partial_\mu \partial_\nu - \eta_{\mu\nu} \partial^2), \\ O_{\mu\nu}^{(3)} &= \frac{2}{3} \xi v_0 [R^\lambda_{(\mu\nu)\tau} + \eta_{\mu\nu} R^\lambda_\tau] y^\tau \partial_\lambda, \\ O_{\mu\nu}^{(4)} &= \frac{1}{3} \xi v_0 R_{\mu\alpha\nu\beta} y^\alpha y^\beta \partial^2, \\ O_{\mu\nu}^{(5)} &= \frac{1}{3} \xi v_0 \eta_{\mu\nu} R_\alpha^{\rho\sigma} y^\alpha y^\beta \partial_\rho \partial_\sigma. \end{aligned} \quad (82)$$

Let us evaluate all the terms of (81), indicating the term in (82) by the left upper index.

The contribution of  $O_{\mu\nu}^{(1)}$  has the form which strongly resembles (78) and can be treated in the same way,

$$\begin{aligned} {}^{(1)}\langle T_{\mu\nu} \rangle_i^1 &= -2\hbar \xi v_0 R_{\mu\nu} \bar{\varepsilon}_0^{(1)} \int d^4y d^4z \int \frac{d^4k}{(2\pi)^4} \frac{e^{ik(y-z)}}{k^2 + 2m^2} \delta^4(x-y) \\ &= -2\hbar \xi v_0 R_{\mu\nu} \bar{\varepsilon}_0^{(1)} \int d^4k \delta^4(k) \frac{e^{ikx}}{k^2 + 2m^2} = -\frac{\hbar \xi v_0}{m^2} R_{\mu\nu} \bar{\varepsilon}_0^{(1)}. \end{aligned} \quad (83)$$

Next, the contribution of  $O_{\mu\nu}^{(2)}$  vanish, for it has the same structure as the flat-space term (78),

$${}^{(2)}\langle T_{\mu\nu} \rangle_i^1 = 0. \quad (84)$$

The contribution of  $O_{\mu\nu}^{(3)}$  can be presented in the form

$${}^{(3)}\langle T_{\mu\nu} \rangle_i^1 = \frac{4\hbar \xi v_0}{3} [R^\lambda_{(\mu\nu)\tau} + \eta_{\mu\nu} R^\lambda_\tau] \bar{\varepsilon}_0^{(1)} I_\lambda^\tau, \quad (85)$$

where

$$I_\lambda^\tau = \int d^4y d^4z \int \frac{d^4k}{(2\pi)^4} \delta^4(x-y) y^\tau \frac{\partial}{\partial y^\lambda} \frac{e^{ik(y-z)}}{k^2 + 2m^2}. \quad (86)$$

The last integral can be calculated by elementary means to give

$$I_\lambda^\tau = \frac{1}{2m^2} \delta_\lambda^\tau \quad (87)$$

and hence, after some small algebra, we obtain

$${}^{(3)}\langle T_{\mu\nu} \rangle_i^1 = -\frac{2\hbar\xi v_0}{3m^2} (R_{\mu\nu} - R\eta_{\mu\nu}) \bar{\varepsilon}_0^{(1)}. \quad (88)$$

The contributions of  $(O)_{\mu\nu}^{(4)}$  and  $(O)_{\mu\nu}^{(5)}$  can be expressed in the form

$${}^{(4,5)}\langle T_{\mu\nu} \rangle_i^1 = \frac{2\hbar\xi v_0}{3} \bar{\varepsilon}_0^{(1)} [R_{\mu\alpha\nu\beta} \eta^{\rho\sigma} - R_{\alpha}{}^{\rho}{}_{\beta}{}^{\sigma} \eta_{\mu\nu}] J_{\rho\sigma},{}^{\alpha\beta}, \quad (89)$$

where

$$J_{\rho\sigma},{}^{\alpha\beta} = \int d^4y d^4z \int \frac{d^4k}{(2\pi)^4} \delta^4(x-y) y^\alpha y^\beta \frac{\partial^2}{\partial y^\rho \partial y^\sigma} \frac{e^{ik(y-z)}}{k^2 + 2m^2}. \quad (90)$$

Taking this integral we obtain

$$J_{\rho\sigma},{}^{\alpha\beta} = \frac{1}{m^2} \delta_{\rho\sigma},{}^{\alpha\beta} \quad \text{where} \quad \delta_{\rho\sigma},{}^{\alpha\beta} = \frac{1}{2} (\delta_\rho^\alpha \delta_\sigma^\beta - \delta_\rho^\beta \delta_\sigma^\alpha). \quad (91)$$

Using this result, after small algebra we arrive at

$${}^{(4,5)}\langle T_{\mu\nu} \rangle_i^1 = \frac{2\hbar\xi v_0}{3m^2} \bar{\varepsilon}_0^{(1)} \left( R_{\mu\nu} + \frac{1}{2} R\eta_{\mu\nu} \right). \quad (92)$$

The total expression can be obtained by replacing the results for all contributions (83), (83), (88) and (92) into (82). After a some algebra we come to the result

$$\langle T_{\mu\nu} \rangle_i^1 = \frac{\hbar\xi v_0}{m^2} \bar{\varepsilon}_0^{(1)} \left( -R_{\mu\nu} + R\eta_{\mu\nu} \right). \quad (93)$$

Obviously, this expression is different from  $G_{\mu\nu}$  and therefore it violates covariance and conservation law. However, if we sum up with the previous result  $\langle T_{\mu\nu} \rangle_v^1$  from Eq. (67), we arrive at the expression which agrees with our expectations,

$$\begin{aligned} \langle T_{\mu\nu} \rangle^1 &= \langle T_{\mu\nu} \rangle_i^1 + \langle \bar{T}_{\mu\nu} \rangle_v^1 \\ &= -2\hbar V_1(v_0) G_{\mu\nu} + \hbar V_0(v_0) g_{\mu\nu} - \frac{\hbar\xi v_0}{m^2} \left( R_{\mu\nu} - \frac{1}{2} Rg_{\mu\nu} \right) \bar{\varepsilon}_0^{(1)}, \\ &= -\hbar \left[ 2V_1(v_0) + \frac{\xi v_0}{m^2} \bar{\varepsilon}_0^{(1)} \right] G_{\mu\nu} + \hbar V_0(v_0) g_{\mu\nu}, \end{aligned} \quad (94)$$

where we finally replaced the flat metric  $\eta_{\mu\nu}$  by the general one  $g_{\mu\nu}$ .

In order to rewrite the quantum contribution in the final form, one needs the expressions for  $V_0(v_0)$ ,  $V_1(v_0)$  and  $\bar{\varepsilon}_0^{(1)}$ . The renormalized and divergent versions of the first two can be obtained from (62) and (63) in the form

$$V_0^{ren}(v_0) = \frac{1}{(4\pi)^2} m^4 \ln\left(\frac{2m^2}{\mu^2}\right), \quad (95)$$

$$V_0^{div}(v_0) = \frac{m^2}{32\pi^2} \left[ 3\Omega^2 - 2m^2 \ln\left(\frac{\Omega^2}{m^2}\right) \right]. \quad (96)$$

and

$$V_1^{ren}(v_0) = -\frac{m^2}{(4\pi)^2} \ln\left(\frac{2m^2}{\mu^2}\right), \quad (97)$$

$$V_1^{div}(v_0) = \frac{1}{32\pi^2} \left( \xi - \frac{1}{6} \right) \left[ -\Omega^2 + 2m^2 \ln\left(\frac{\Omega^2}{m^2}\right) \right]. \quad (98)$$

Taking into account in (94) the expressions for  $\bar{\varepsilon}_0^{(1)}$  derived in Eqs. (115) and (116) of Appendix B, we arrive at the final result for quantum contributions to EMT,

$$\langle T_{\mu\nu} \rangle_{ren} = \frac{\hbar m^4}{(4\pi)^2} \ln\left(\frac{2m^2}{\mu^2}\right) g_{\mu\nu} - \frac{m^2}{(4\pi)^2} \left[ 2(1 + 3\xi) \ln\left(\frac{2m^2}{\mu^2}\right) + 3\xi \right] G_{\mu\nu} \quad (99)$$

for the renormalized expression and

$$\begin{aligned}
\langle T_{\mu\nu} \rangle_{div} &= \frac{\hbar m^2}{32\pi^2} \left[ 3\Omega^2 - 2m^2 a\left(\frac{\Omega^2}{m^2}\right) \right] g_{\mu\nu} \\
&+ \frac{\hbar}{16\pi^2} \left( 4\xi - \frac{1}{6} \right) \left\{ \Omega^2 - 2m^2 \ln \frac{\Omega^2}{m^2} \right\} G_{\mu\nu}
\end{aligned} \tag{100}$$

for the divergent one.

## VII. CONCLUSIONS

We have considered several aspects of the Energy-Momentum Tensor (EMT) of vacuum in curved space-time. A naive calculation using a momentum cut-off produces a result which apparently violates general covariance. It was noticed long ago that this is the effect of the non-covariant cut-off scheme and therefore can be hardly regarded to be a physical feature of the theory. The two questions naturally arise in this respect, namely whether it is possible to introduce a cut-off on a covariant way and whether it is possible to have a non-trivial quantum contributions to the Energy-Momentum Tensor of Vacuum.

We have addressed the first and in part the second issue on the basis of effective action method. In both cases the output of our investigation was perfectly consistent with the general expectations based on the known structure of renormalization in curved space-time and conservation law for the EMT. The calculations in the theory with SSB have shown some new term which was unnoticed until now. At the same time, after performing explicit calculations of this term we have found that the final results, Eqs. (99) and (100), have the usual form and that the quantum effects always lead only to the renormalization of the inverse Newton constant and cosmological constant in Eqs. (50) and in the relations such as (51) and (52).

It is important to note that our calculations can not be interpreted as a no-go theorem for a non-trivial quantum contributions to the low-energy sector of the gravitational action. As it was previously explained in [13], the chance to meet such corrections exists, but this can be verified only in the framework of some qualitatively new mathematical tool which should not be based on the perturbative expansion in curvatures. The linear in curvature approximation which was adopted here does not provide any information about these type of corrections. However, it was definitely worthwhile to check that the standard considerations really work for the non-trivial physical situations such as gravity combined with SSB.



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### Appendix A. Local momentum representation

The calculations in Sect. 6 were done using Riemann normal coordinates (see, e.g., [41] for introduction) and the local momentum representation technique (see, e.g., [39, 40]). In this Appendix we present some necessary elements of these tools and also derive the operator  $g_{\mu\nu}\square - \nabla_\mu\nabla_\nu$  because it is related to relatively trivial calculations.

The normal coordinates expansion performs around one special point  $P$  (that is related methods work well only for deriving local quantities), where metric is supposed to be flat Minkowski one. However, the derivatives of the metric, starting from the second one, are of course non-zero. The expression for the metric is

$$g_{\mu\nu} = \eta_{\mu\nu} - \frac{1}{3}R_{\mu\alpha\nu\beta}y^\alpha y^\beta + \dots \quad (101)$$

Here and below all components of the curvature tensor correspond to the point  $P$ , also in (101) we have omitted the higher order terms in curvature tensors and their derivatives. Furthermore  $y^\alpha$  represent deviation from the point  $P$ , such that all partial derivatives below are taken with respect to  $y^\alpha$ . It is fairly easy to derive, using (101), the following expansions:

$$\begin{aligned} g^{\mu\nu} &= \eta^{\mu\nu} + \frac{1}{3}R^\mu{}_\alpha{}^\nu{}_\beta y^\alpha y^\beta, \\ \Gamma_{\mu\nu}^\lambda &= -\frac{2}{3}R^\lambda{}_{(\mu\nu)}{}^\tau y^\tau. \end{aligned} \quad (102)$$

Then for the two covariant derivatives acting on scalar we obtain

$$\nabla_\mu\nabla_\nu = \partial_\mu\partial_\nu + \frac{2}{3}R^\lambda{}_{(\mu\nu)}{}^\tau y^\tau\partial_\lambda. \quad (103)$$

Making contraction with

$$g^{\mu\nu} = \eta^{\mu\nu} + \frac{1}{3}R^\mu{}_\lambda{}^\nu{}_\beta y^\alpha y^\beta, \quad (104)$$

we get

$$\square = g^{\mu\nu} \nabla_\mu \nabla_\nu = \partial^2 - \frac{2}{3} R^\lambda_\tau y^\tau \partial_\lambda + \frac{1}{3} R^\mu_{\alpha}{}^\nu_{\beta} y^\alpha y^\beta \partial_\mu \partial_\nu, \quad (105)$$

where  $\partial^2 = \eta^{\mu\nu} \partial_\mu \partial_\nu$ . Finally, the first operator of our interest is

$$\begin{aligned} \nabla_\mu \nabla_\nu - g_{\mu\nu} \square &= \partial_\mu \partial_\nu + \frac{2}{3} \dot{R}^\lambda_{(\mu\nu)} y^\tau \partial_\lambda - \eta_{\mu\nu} \partial^2 + \frac{2}{3} \eta_{\mu\nu} \dot{R}^\lambda_\tau y^\tau \partial_\lambda \\ &\quad - \frac{1}{3} \eta_{\mu\nu} \dot{R}^\rho_{\alpha}{}^\sigma_{\beta} y^\alpha y^\beta \partial_\rho \partial_\sigma - \frac{1}{3} \dot{R}_{\mu\alpha\nu\beta} y^\alpha y^\beta \partial^2. \end{aligned} \quad (106)$$

The next operator we are interested in is the propagator of scalar field. Direct calculations using (105) (see, e.g., [39, 40]) lead to the relevant expression for the propagator of the field of the mass  $m$  in the linear in curvature approximation,

$$\bar{G}(y) = \int \frac{d^4 k}{(2\pi)^4} e^{iky} \left[ \frac{1}{k^2 + m^2} - \left( \xi - \frac{1}{6} \right) \frac{R}{(k^2 + m^2)^2} \right], \quad (107)$$

where we already assumed Wick rotation to Euclidean space. The Eq. (107) was recently used in [32] to derive the effective potential of scalar field in the momentum cut-off regularization. Due to the use of local momentum representation (107) the result has covariant form, despite the naive application of the cut-off scheme is supposed to break down even Lorentz invariance.

## Appendix B. Effective equations of motion

Here we present the effective equations of motion on the  $\mathcal{O}(R)$ -approximation, when the effective action is reduced to (61). By using Eqs. (61) and (66) we obtain

$$\bar{\varepsilon}_0^{(1)} = - \frac{\partial \bar{V}_0}{\partial \phi} \Big|_{v_0} \quad (108)$$

and

$$\bar{\varepsilon}_1^{(1)} = - \frac{\partial^2 \bar{V}_0}{\partial \phi^2} \Big|_{v_0} \cdot v_1 - R \frac{\partial \bar{V}_1}{\partial \phi} \Big|_{v_0}. \quad (109)$$

Let us denote the curvature-independent and mass-independent part of classical potential as

$$V = V(\phi) = \frac{\lambda}{4} \phi^4. \quad (110)$$

From the quantities  $\bar{V}_0$  and  $\bar{V}_1$  given by (63) and (64) one can easily get

$$\begin{aligned} \frac{\partial \bar{V}_0^{div}}{\partial \phi} &= \frac{1}{32\pi^2} \left[ \Omega^2 V''' - (V'' - m^2) V''' \ln \frac{\Omega^2}{m^2} \right], \\ \frac{\partial^2 \bar{V}_0^{div}}{\partial \phi^2} &= \frac{1}{32\pi^2} \left[ \Omega^2 V'''' - (V''')^2 \ln \frac{\Omega^2}{m^2} - V'''' (V'' - m^2) \ln \frac{\Omega^2}{m^2} \right], \\ \frac{\partial \bar{V}_1^{div}}{\partial \phi} &= \frac{1}{32\pi^2} \left( \xi - \frac{1}{6} \right) V''' \ln \frac{\Omega^2}{m^2}. \end{aligned} \quad (111)$$

Furthermore, from  $\bar{V}_{ren}^{(0)}$  and  $\bar{V}_{ren}^{(1)}$  in (62) we obtain

$$\begin{aligned}\frac{\partial \bar{V}_0^{ren}}{\partial \phi} &= \frac{1}{32\pi^2} (V'' - m^2) V''' \left[ \ln \left( \frac{V'' - m^2}{\mu^2} \right) + \frac{1}{2} \right], \\ \frac{\partial^2 \bar{V}_0^{ren}}{\partial \phi^2} &= \frac{1}{32\pi^2} \left\{ [(V''')^2 + (V'' - m^2) V'''] \left[ \ln \left( \frac{V'' - m^2}{\mu^2} \right) + \frac{1}{2} \right] + (V''')^2 \right\}, \\ \frac{\partial \bar{V}_1^{ren}}{\partial \phi} &= -\frac{1}{32\pi^2} \left( \xi - \frac{1}{6} \right) V''' \left[ \ln \left( \frac{V'' - m^2}{\mu^2} \right) + 1 \right].\end{aligned}\quad (112)$$

Next, we calculate the on-shell expressions by replacing  $\phi \rightarrow \phi_{0c}$  and  $\lambda v_0^2 = 6m^2$ , in the form

$$\begin{aligned}\left. \frac{\partial \bar{V}_0^{div}}{\partial \phi} \right|_{v_0} &= \frac{1}{32\pi^2} \left[ \lambda v_0 \Omega^2 - 2\lambda m^2 v_0 \ln \frac{\Omega^2}{m^2} \right], \\ \left. \frac{\partial \bar{V}_1^{div}}{\partial \phi} \right|_{v_0} &= \frac{1}{32\pi^2} \left( \xi - \frac{1}{6} \right) \lambda v_0 \ln \frac{\Omega^2}{m^2}, \\ \left. \frac{\partial^2 \bar{V}_0^{div}}{\partial \phi^2} \right|_{v_0} &= \frac{1}{32\pi^2} \left[ \lambda \Omega^2 - 8\lambda m^2 \ln \frac{\Omega^2}{m^2} \right].\end{aligned}\quad (113)$$

Similarly, the analogous renormalized on-shell expressions are

$$\begin{aligned}\left. \frac{\partial \bar{V}_0^{ren}}{\partial \phi} \right|_{v_0} &= \frac{1}{16\pi^2} \lambda m^2 v_0 \left[ \ln \left( \frac{2m^2}{\mu^2} \right) + \frac{1}{2} \right], \\ \left. \frac{\partial^2 \bar{V}_0^{ren}}{\partial \phi^2} \right|_{v_0} &= \frac{\lambda m^2}{16\pi^2} \left[ 4 \ln \left( \frac{2m^2}{\mu^2} \right) + 5 \right], \\ \left. \frac{\partial \bar{V}_1^{ren}}{\partial \phi} \right|_{v_0} &= -\frac{\lambda v_0}{32\pi^2} \left( \xi - \frac{1}{6} \right) \left[ \ln \left( \frac{2m^2}{\mu^2} \right) + 1 \right].\end{aligned}\quad (114)$$

At this point we can derive the elements of equations of motion,

$$\bar{\varepsilon}_{0,div}^{(1)} = \left. \frac{\partial \bar{V}_{div}^{(0)}}{\partial \phi} \right|_{v_0} = -\frac{\lambda v_0}{32\pi^2} \Omega^2 + \frac{\lambda m^2 v_0}{16\pi^2} \ln \frac{\Omega^2}{m^2}, \quad (115)$$

$$\bar{\varepsilon}_{0,ren}^{(1)} = \left. \frac{\partial \bar{V}_{ren}^{(0)}}{\partial \phi} \right|_{v_0} = -\frac{1}{16\pi^2} \lambda m^2 v_0 \left[ \ln \left( \frac{2m^2}{\mu^2} \right) + \frac{1}{2} \right], \quad (116)$$

$$\begin{aligned}\bar{\varepsilon}_{1,div}^{(1)} &= -\left. \frac{\partial^2 \bar{V}_0^{div}}{\partial \phi^2} \right|_{v_0} \cdot v_1 - R \left. \frac{\partial \bar{V}_1^{div}}{\partial \phi} \right|_{v_0} \\ &= -\frac{3\xi}{32\pi^2 v_0} \Omega^2 R + \frac{\lambda v_0}{32\pi^2} \left( 3\xi + \frac{1}{6} \right) R \ln \frac{\Omega^2}{m^2},\end{aligned}\quad (117)$$

$$\begin{aligned}\bar{\varepsilon}_{1,ren}^{(1)} &= -\left. \frac{\partial^2 \bar{V}_0^{ren}}{\partial \phi^2} \right|_{v_0} \cdot v_1 - R \left. \frac{\partial \bar{V}_1^{ren}}{\partial \phi} \right|_{v_0} \\ &= -\frac{\lambda v_0}{32\pi^2} \left( 3\xi + \frac{1}{6} \right) R \ln \left( \frac{2m^2}{\mu^2} \right) - \left( 4\xi + \frac{1}{6} \right) \frac{\lambda v_0}{32\pi^2} R.\end{aligned}\quad (118)$$

The last observation is that, as we have already mentioned in the main text, all these expressions must be treated as constants in the given approximation.

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